THE SCOTS COLLEGE

Year 12 Mathematics Extension 2
Assessment 1
February 2007

GENERAL INSTRUCTIONS

- Working time - 50 minutes
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question

TOTAL MARKS: 40

WEIGHTING: 10%

- Attempt all questions

OUTCOMES

QUESTION/ OUTCOME	A student combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions	A student uses the relationship between algebraic and geometric representations of complex numbers
Question 1	/10	
Question 2		/30
TOTAL PERCENTAGE	/40	

QUESTION 1 **[10 MARKS]** **MARKS**

(i) Sketch the curve $f(x)=x^2-2x-3$ on the number plane provided at back. **[2]**

(ii) Sketch $y=\frac{1}{f(x)}$ on a separate diagram **[2]**

(iii) Sketch $y=\sqrt{f(x)}$ on a separate number plane. **[2]**

(iv) Sketch $y=f(|x|)$ on a separate diagram **[2]**

(v) Sketch $\text{Ln}\, f(x)$ on the number plane displaying the graph of $y=f(x)$. **[2]**

QUESTION 2 **[10 MARKS]**

The points A and B on the Argand plane provided represent the complex numbers z_1 and z_2 respectively.

Mark the position of the following complex numbers with the letter indicated, on the number plane provided at back:

(i) C = z_1+z_2 **[2]**

(ii) D = z_1-z_2 **[2]**

(iii) E = iz_2 **[2]**

(iv) F = z_1z_2 **[2]**

(v) G = $\overline{z_2-z}$

QUESTION 3 **[10 MARKS]** **MARKS**

Sketch the graphs of the following on the Argand plane provided.

a. **(i)** $0 \le \text{Im}|z| \le \sqrt{3}$, $1 \le \text{Re}(z) \le 3$ **[2]**

(ii) Determine all possible values of $\arg(z)$. **[1]**

b. **(i)** Sketch $|z-2+2i| \le 2$ **[2]**

(ii) Hence or otherwise determine the maximum value of $|z|$. **[2]**

c. **(i)** Sketch the locus represented by $Arg\left(\frac{z-1}{z-i}\right) = \frac{\pi}{4}$ **[2]**

(ii) Describe this locus geometrically. **[1]**

QUESTION 4 **[10 MARKS]** **MARKS**

a. **(i)** Express $z = 1 + \sqrt{3}i$ on the form $R\ cis\ \theta$. **[2]**

(ii) Hence verify the result $|z|^2 = z\overline{z}$ **[1]**

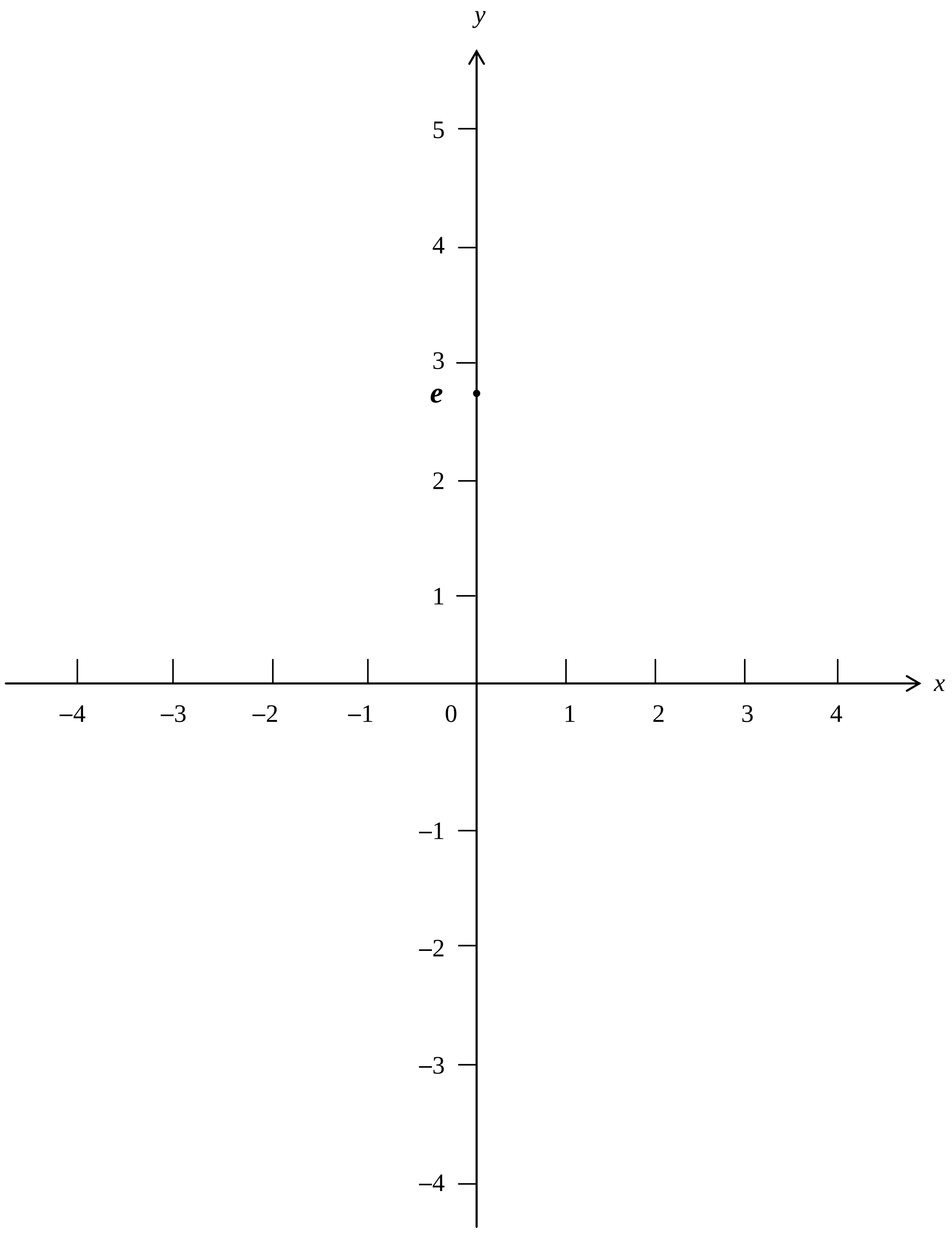
(iii) Find the square roots of z. **[2]**

b. **(i)** Find the five fifth roots of unity. **[3]**

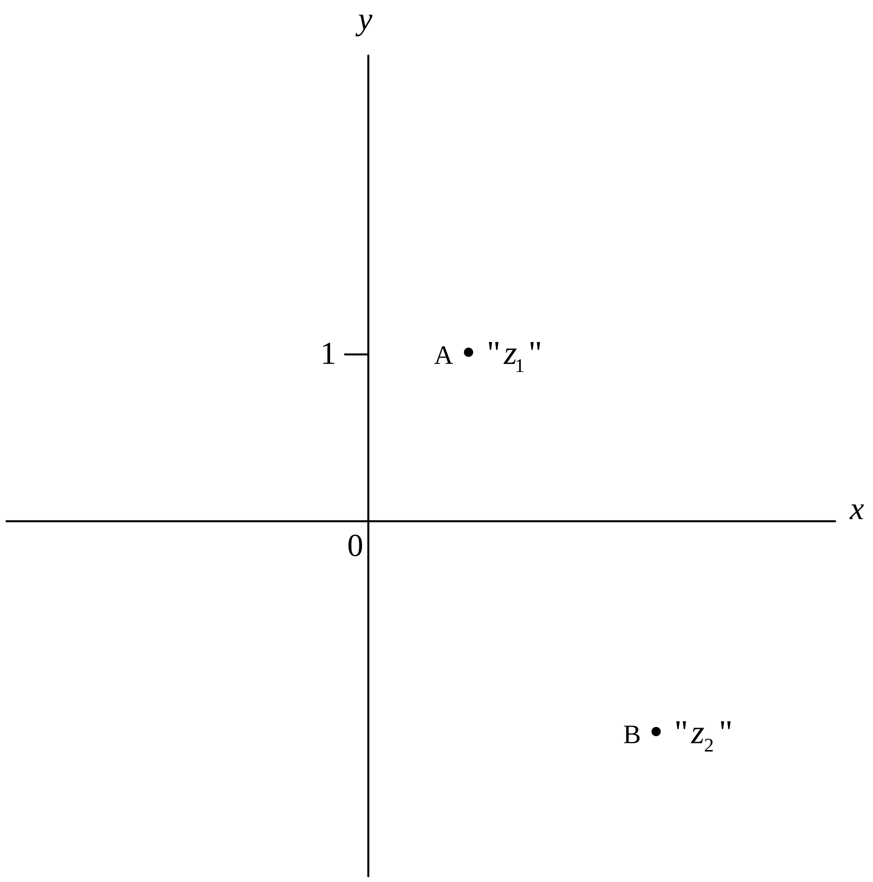
(ii) By considering $z^5 = 1$ show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ **[2]**

QUESTION 1　　**ANSWER SHEET**

Parts (i) and (v) <u>ONLY</u>



QUESTION 2 ANSWER SHEET



SOLUTIONS Yr 12 EXT 2
ASSESS. I FEB. 2007

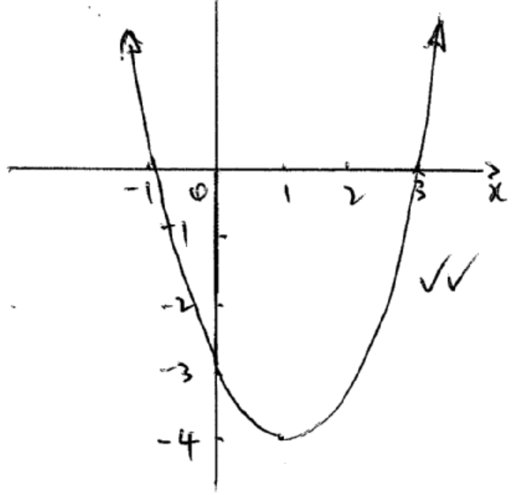
Q1.

(i) $f(x) = x^2 - 2x - 3$

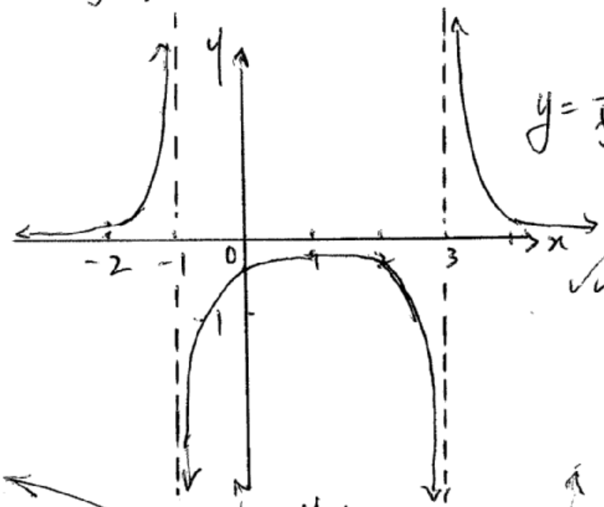
$= (x-3)(x+1)$.

$f(0) = -3$

$f(x) = 0$ where $x = -1$ or 3.

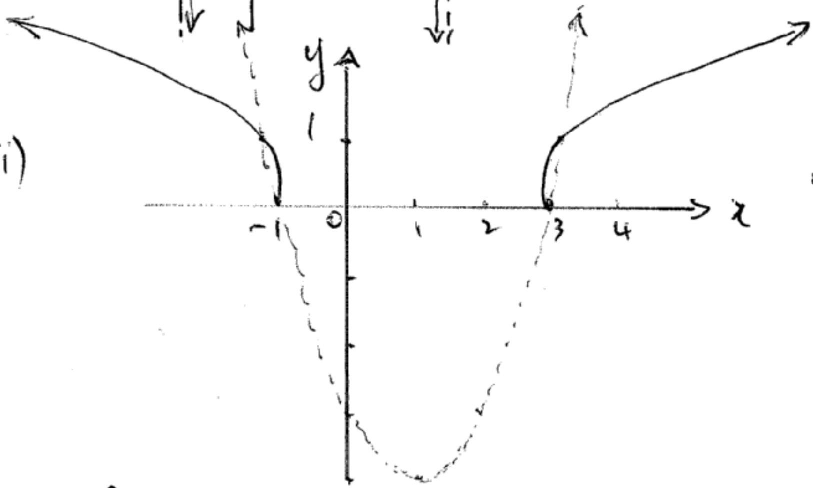


(ii)



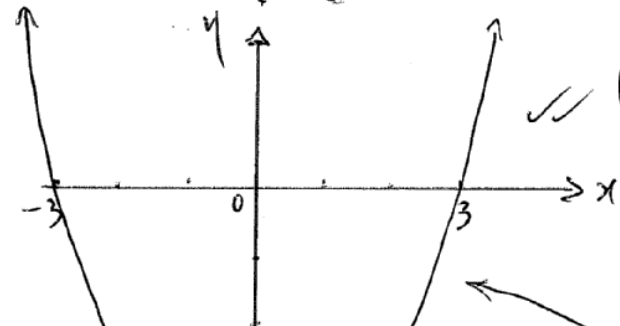
$y = \frac{1}{f(x)}$.

(iii)



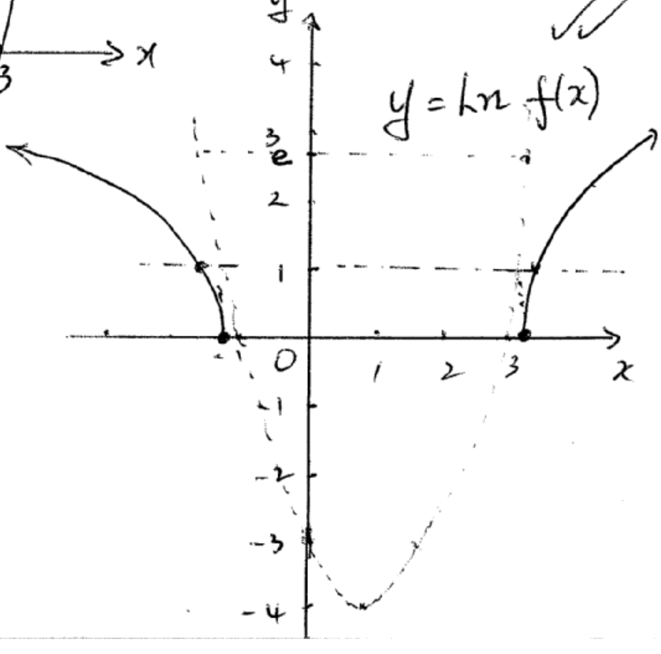
$y = \sqrt{f(x)}$.

(iv)



$y = f(|x|)$

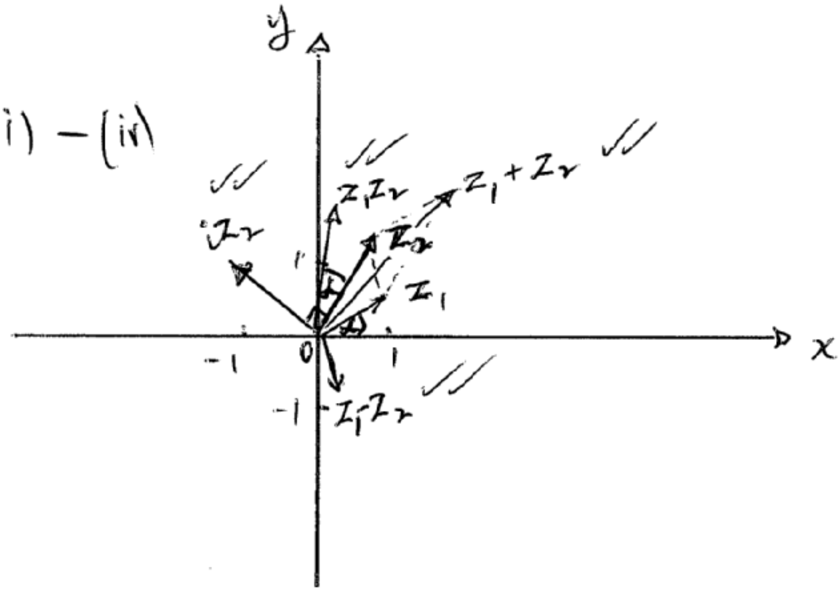
(v)



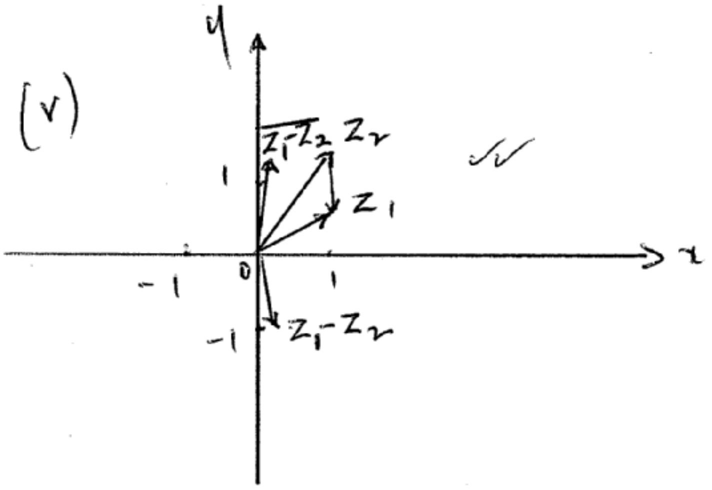
$y = \ln f(x)$

Q2

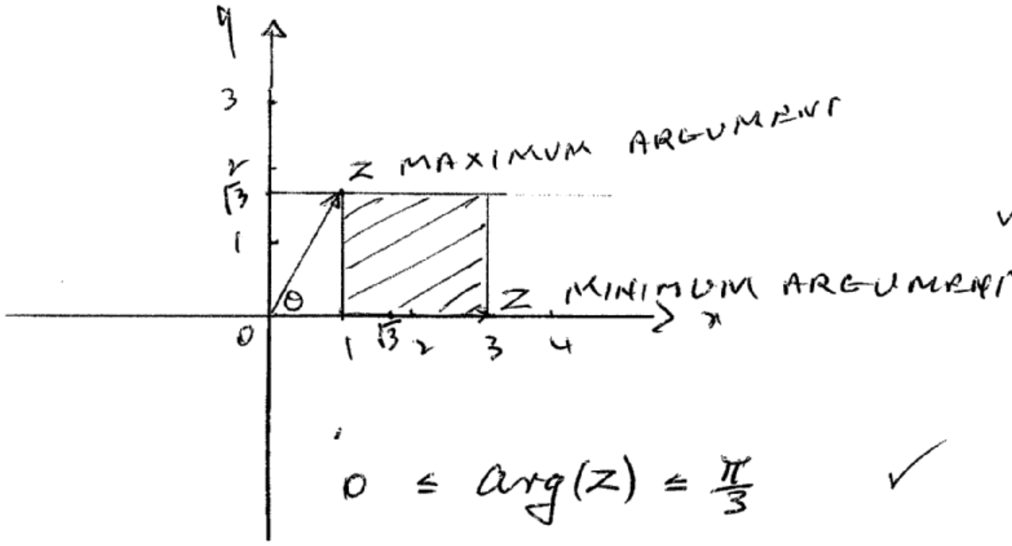
(i) – (iv)



(v)



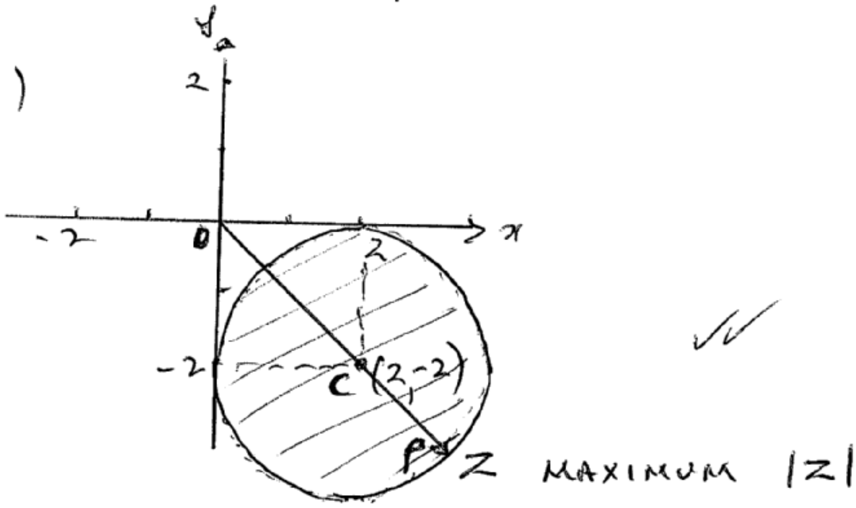
Q3 (a) (i)



(ii) $0 \le \arg(z) \le \frac{\pi}{3}$ ✓

(b) $|z-(2-2i)| \le 2$

(i)



(ii) $OC = 2\sqrt{2}$

$CP = 2$ (Radius)

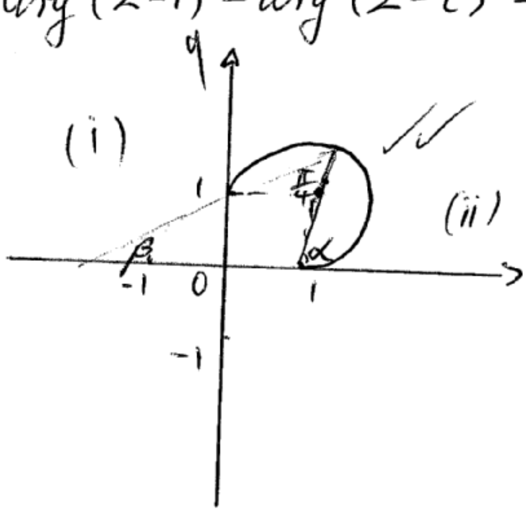
So $OP = 2 + 2\sqrt{2}$

$|z| = 2(1+\sqrt{2})$

(c) $\arg(z-1) - \arg(z-i) = \frac{\pi}{4}$ ⟶ Let $\alpha = \arg(z-1)$ and $\beta = \arg(z-i)$

So $\alpha - \beta = \frac{\pi}{4}$.

(i)



(ii) Major arc of a circle centre $C(1,1)$ and radius 1 unit terminating at $(0,1)$ and $(1,0)$.

Q4 (a) (i) $z = 1+\sqrt{3}i$

$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ ✓✓

(ii) $|z| = 2$

$|z|^2 = 4$.

$z\bar{z} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$= 4\left(\cos^2\frac{\pi}{3} - i^2\sin^2\frac{\pi}{3}\right)$

$= 4\left(\cos^2\frac{\pi}{3} + \sin^2\frac{\pi}{3}\right)$

$= 4$ ✓

$= |z|^2$

(iii) For $z^2 = 1+\sqrt{3}i$

$|z^2| = |1+\sqrt{3}i| = 2$

$|z| = \sqrt{2}$ and so $z = \sqrt{2}\,\text{cis}\,\theta$

Now $(\sqrt{2}\,\text{cis}\,\theta)^2 = 2\,\text{cis}\,\frac{\pi}{3}$ ✓

$2\,\text{cis}\,2\theta = 2\,\text{cis}\,\frac{\pi}{3}$

$\cos 2\theta + i\sin 2\theta = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\cos 2\theta = \frac{1}{2}$, $\sin 2\theta = \frac{\sqrt{3}}{2}$

$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \ldots$, $2\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3} \ldots$

So $2\theta = \frac{\pi}{3}, \frac{7\pi}{3}$ ✓

$\theta = \frac{\pi}{6}, \frac{7\pi}{6}$

and $z_1 = \sqrt{2}\,\text{cis}\,\frac{\pi}{6}$, $z_2 = \sqrt{2}\,\text{cis}\,\frac{7\pi}{6}$.

(b) (i) Let $z = \cos\theta + i\sin\theta$ be a root of unity

So $(\cos\theta + i\sin\theta)^5 = 1$ ✓

$\cos 5\theta + i\sin 5\theta = 1$

$\cos 5\theta = 1$ ✓

$5\theta = 0^\circ, 2\pi, 4\pi, 6\pi, 8\pi$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

Roots are: $z_1 = 1$, $z_2 = \text{cis}\left(-\frac{2\pi}{5}\right)$, $z_3 = \text{cis}\left(-\frac{4\pi}{5}\right)$, ✓

$z_4 = \text{cis}\,\frac{2\pi}{5}$, $z_5 = \text{cis}\,\frac{4\pi}{5}$.

(ii) Sum of roots is $z_1 + (z_2+z_4) + (z_3+z_5)$ – Pairing conjugates ✓

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$ (From $z^5 - 1 = 0$)

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ ✓